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Semi-Implicit Time Integration in Limited Area Model

Isidore Halberstam

ST Systems Corporation
109 Massachusetts Avenue
Lexington, MA 02173

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Hanscom Air Force Base, Massachusetts 01731-5000

This technical report has been reviewed and is approved for publication.

CHIEN-HSIUNG YANG
Contract Manager

DONALD A. CHISHOLM, Chief
Atmospheric Prediction Branch

FOR THE COMMANDER

ROBERT A. McCLATCHEY, Director
Atmospheric Sciences Division

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An analysis of the Australian Bureau of Meteorology Research Centre (BMRC) semi-implicit scheme is presented. The scheme is discussed for possible adaptation to the Relocatable Limited-Area Model (RLAM). It is found that the scheme has certain peculiarities which make it difficult to blend with the other versions of RLAM. It is shown, for example, that a small change in representing vertical differencing can have a severely damaging impact on the surface pressure forecast.

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1. Introduction

As numerical models of the atmosphere become more finely resolved, restraints on the size of time steps become increasingly difficult to deal with. These restraints in explicit time differencing are due to the well-known Courant-Friedrich-Lewy (CFL) conditions which limit the time step to the ratio of the spatial increment to the phase velocity of the predicted variable. Thus, as the spatial increment is decreased, the time-step must also be reduced. Implicit time schemes avoid these restrictions. It has been shown that for linear systems, time schemes where the time dependent variable in the numerical approximation to the analytic equation is given at the predicted time-step (thereby requiring an "implicit" solution) are not subject to CFL limitations. In fact, an Euler-backward scheme where all references to the variable are given at the predicted time step can be shown to be absolutely stable no matter how large the time increment relative to the space increment. This characteristic makes implicit schemes appealing for application to numerical weather prediction. But there are other factors involved in numerical weather prediction. For one thing, the equations are non-linear and no one can guarantee stability, even with implicit schemes, for non-linear formulations. Non-linearity also poses practical problems in solving for the predicted time step. Whereas with linear equations, the terms involving the predicted variable can easily be separated from terms involving known quantities, with non-linear configurations, the separation and hence the solution methodology may not be so straightforward. Some of these impediments can be overcome by introducing semi-implicit schemes.

Semi-implicit schemes derive their label from the fact that only portions of the equation (generally, the linear portions) are cast in implicit formulations. The rest of the equation is left as an explicit expression of known quantities. In this form the scheme is not absolutely stable, and the size of the time-step does play a role. Kurihara (1965), in fact, demonstrated that semi-implicit schemes are weakly unstable and may not be able to resolve short waves efficiently, if the time step is excessive. The problem is magnified when physical processes are included.

In choosing a semi-implicit scheme for the relocatable limited-area model (RLAM) developed by ST Systems, Inc., as described by Tung et al., (1988), in support of the global modelling effort at the Air Force Geophysics Laboratory (AFGL), several different models were examined. These included AFGL's global spectral model (gsm) itself, the baroclinic model of the Met Service of Canada (Robert, et al., 1972), and the regional model of the Australian

Numerical Meteorological Research Centre, now Bureau of Meteorology Research Centre (BMRC) as detailed by McGregor, et al., (1978). The methodology in all three models is the same. A single equation in one variable is solved. The solution is then used to solve for the other variables expressed in terms of the first. In the case of the gsm, an updated value of divergence is derived first; afterwards vorticity, surface pressure, and temperature are forecast based on the new value of divergence. The Canadian model first solves for a variable which combines geopotential and surface pressure and derives all others from it, while the Australians first solve for temperature. In all cases, the single equation in one variable is a Helmholtz equation, where the right hand side contains all the non-linear terms evaluated from known quantities at the present or past time steps. The gsm and the Canadian model approach were rejected in favor of the Australian model because of greater compatibility with RLAM. The gsm, being a spectral model, can carry vorticity and divergence as principal dependent variables that are solved by the model's algorithm and then transformed diagnostically into wind velocity components. The RLAM, being a primitive equations, finite-difference model, as shall be seen in the next section, carries the velocity components themselves as principal variables. The Canadian model does employ finite differencing, but its vertical staggering of variables is dissimilar to the gsm and RLAM. It may have been possible to reconfigure the

Canadian model for use by RLAM, but since BMRC's model was completely compatible with RLAM, the path of least resistance was chosen. The construction of the BMRC model is theoretically consistent with RLAM's and and as such, it was expected that adapting to RLAM would pose no problem. Unfortunately, as one discovers quite often in numerical modeling, theoretical consistency does not always lead to applicability. In the ensuing sections, difficulties in adapting the BMRC scheme to RLAM will be presented.

2. Review of RLAM

RLAM has been documented in various stages of its development, notably in Tung et al., (1988) and Halberstam (1988). Briefly, RLAM was designed to have multiple modules so that forecasts can be made using any combination from a number of differencing schemes both spatial and temporal. The spatial choices consist of second or fourth order differencing. Fourth order compact differencing had originally been included but has been dropped because it did not offer much in the way of advantage over the regular fourth order scheme and required specifications along the boundaries which were just as troublesome to deal with as the regular fourth order. Along with the horizontal differencing, one can specify the frequency and order of smoothing along the lateral boundaries and/or over the entire field. Second or fourth order diffusion can also be specified with a variable (in space) diffusion coefficient. Physical parameterizations are optional. The model allows invoking one or all of bulk boundary layer specifications, large-scale precipitation, and a Kuo-type convection.

The lateral boundary conditions are also subject to the selective process. At present, one can select among a Davies-type (1976), Perkey-Kreitsberg (1976), or Orlanski-type (1976) boundary specification. The selection of lateral boundary specifications is crucial for limited-area models. Great care must be observed in choosing boundary conditions which are compatible with the spatial and temporal differencing schemes. In any case, because gsm data form some part of the boundary specification, it is not conceivable that the limited-area model can depart too far from the gsm without causing major disruptions. Thus, one is faced with the disparate goals of expecting the limited-area model to improve over the global model in the interior on the one hand while also expecting the global model to provide useful boundary conditions for the limited-area model. Because in practice the model forecasts tend to drift apart, modelers have had to devise boundary specifications of the form mentioned here, where either a slow blending of large-scale and interior boundary conditions occurs or internal waves are allowed to exit the domain without much disruption while exterior waves are allowed to enter if conditions of this sort prevail. It also requires that the forecast be limited in time so that the boundaries do not swamp the interior completely.

For time schemes, RLAM originally included either a standard explicit leap frog scheme or a Brown-Campana pressure-gradient averaging scheme. The latter is still explicit but the time-averaging affords more liberal time steps. Because these time schemes are explicit, the time steps have to be shortened as the spatial resolution is increased. Thus, for high-resolution integrations, a semi-implicit or equivalent scheme was required in order to keep the model tenable and economically feasible. As already mentioned, the BMRC scheme was selected because of its compatibility with RLAM. The following section summarizes the theoretical basis for the scheme.

3. The BMRC semi-implicit scheme

Starting with a set of primitive equations, the BMRC model consolidates the whole set into a single equation in one variable. The original equations are

a. $$\frac{\partial u}{\partial t} = -m_x m_y \left[u \frac{\partial}{\partial x} \left(\frac{u}{m_y} \right) + v \frac{\partial}{\partial y} \left(\frac{u}{m_x} \right) \right] - \dot{\sigma} \frac{\partial u}{\partial \sigma} - m_x \left[\frac{\partial \phi}{\partial x} + RT \frac{\partial \ln p_*}{\partial x} \right] + fv + F_x$$

b. $$\frac{\partial v}{\partial t} = -m_x m_y \left[u \frac{\partial}{\partial x} \left(\frac{v}{m_y} \right) + v \frac{\partial}{\partial y} \left(\frac{v}{m_x} \right) \right] - \dot{\sigma} \frac{\partial v}{\partial \sigma} -$$

$$m_y \left[\frac{\partial \phi}{\partial y} + RT \frac{\partial \ln p_*}{\partial y} \right] - fu + F_y$$

c. $$\frac{\partial T}{\partial t} - \frac{T}{\pi} \frac{\partial \pi}{\partial t} = -m_x u \frac{\partial T}{\partial x} - m_y v \frac{\partial T}{\partial y} + \frac{T}{\pi} \left[m_x u \frac{\partial \pi}{\partial x} + m_y v \frac{\partial \pi}{\partial y} \right] -$$

$$\dot{\sigma} \frac{\partial T}{\partial \sigma} + \frac{\dot{\sigma}}{\pi} T \frac{\partial \pi}{\partial \sigma} + H$$

(1)

d. $$\frac{\partial \ln p_*}{\partial t} = -m_x m_y \left[\frac{u}{m_y} \frac{\partial \ln p_*}{\partial x} + \frac{v}{m} \frac{\partial \ln p_*}{\partial y} \right] -$$

$$m_x m_y \left[\frac{\partial \left(\frac{u}{m_y} \right)}{\partial x} + \frac{\partial \left(\frac{v}{m_x} \right)}{\partial y} \right] - \frac{\partial \dot{\sigma}}{\partial \sigma}$$

e. $$\frac{\partial \phi}{\partial \sigma} = -\frac{RT}{\sigma}$$

f. $$\frac{\partial q}{\partial t} = - m_x u \frac{\partial q}{\partial x} - m_y v \frac{\partial q}{\partial y} - \dot{\sigma} \frac{\partial q}{\partial \sigma} + S$$

T - temperature

u - Eastward wind velocity component

v - Northward wind velocity component

q - specific humidity

p_* - surface pressure

ϕ - geopotential

t - time

x, y - horizontal map coordinates

σ - vertical coordinate = $\frac{p}{p_*}$ (p is pressure)

f - coriolis parameter

m_x, m_y - horizontal map factors

π - Exner function $(\frac{p}{p_o})^{\kappa}; \kappa=\frac{R}{c_p}$

R - Ideal gas constant

c_p - heat capacity of air at constant pressure

$\dot{\sigma}$ - vertical velocity $\equiv \frac{d\sigma}{dt}$

F_x, F_y - horizontal friction

H - diabatic heat sources or sinks

S - moisture sources or sinks

Using the following notation for time averaging, $\frac{A^{t+1} + A^{t-1}}{2} \equiv \bar{A}^{\tau}$, allows us to write the discrete approximation to the time derivative $\frac{\partial A}{\partial t}$ as $(\bar{A}^{\tau} - A^{t-1})/\Delta t$, where A^{t+1} is the variable A at time $t+\Delta t$ and A^{t-1} is A at time $t-\Delta t$. This centered differencing along with the accompanying right-hand side containing terms in A^t, requires specification at three time levels. We also separate the temperature, T, into a climatological component, T_o, which depends only on the vertical coordinate, σ, and a spatially and temporally varying component T'. Given these specifications we can approximate (1) as

$$\text{a.} \quad \bar{u}^{\tau} + \Delta t m_x \left[\frac{\partial(\bar{\phi}^{\tau} - \phi_*)}{\partial x} + RT_o \frac{\partial \ln \bar{p}_*^{\tau}}{\partial x} \right] = a + \Delta t m_x RT_o \frac{\partial \ln p_*^{t-1}}{\partial x}$$

$$\text{b.}\quad \bar{v}^{\tau}+\Delta t m_y\left[\frac{\partial(\bar{\phi}^{\tau}-\phi_*)}{\partial y} + RT_o\frac{\partial \ln \bar{p}^{\tau}_*}{\partial y}\right] = b + \Delta t m_y RT_o\frac{\partial \ln p^{t-1}_*}{\partial y}$$

(2)

$$\text{c.}\quad \bar{T}^{\tau}-\Delta t\left(\frac{\kappa T_o}{\sigma} - \frac{\partial T_o}{\partial \sigma}\right)\bar{W}^{\tau} - \Delta t\sigma\frac{\partial T_o}{\partial \sigma}\,\bar{W}^{\tau}_* = c$$

$$\text{d.}\quad \frac{\partial \bar{W}^{\tau}}{\partial \sigma} + m_x m_y\left[\frac{\partial \dfrac{\bar{u}^{t}}{m_y}}{\partial x} + \frac{\partial \dfrac{\bar{v}^{\tau}}{m_x}}{\partial y}\right] = -m_x m_y\left[\frac{u}{m_y}\,\frac{\partial \ln p_*}{\partial x} + \frac{v}{m_x}\,\frac{\partial \ln p_*}{\partial y}\right],$$

where the new variable W is equal to $\dot{\sigma} + \sigma\,\frac{\partial \ln p_*}{\partial t}$ and because at the surface $\sigma = 1$, and $\dot{\sigma} = 0$, $W_* = \frac{\partial \ln p_*}{\partial t}$, or, in finite difference specification, $\Delta t\ \bar{W}^{\tau}_* = \ln \bar{p}^{\tau}_* - \ln p^{t-1}_*$. The moisture equation is solved explicitly using the updated values of u and v for advection and is therefore not involved in these manipulations and hence not listed in (2). a, b, and c account for the non-linear portions of the momentum and thermodynamic equations and are given as

$$\text{A.}\quad a = u^{t-1} - \Delta t m_x m_y\left[u\frac{\partial}{\partial x}\left(\frac{u}{m_y}\right) + v\,\frac{\partial}{\partial y}\left(\frac{u}{m_x}\right)\right] - \Delta t(\dot{\sigma}\,\frac{\partial u}{\partial \sigma} - fv + F_x)$$

$$-\Delta t m_x\left[RT'\frac{\partial \ln p_*}{\partial x} + RT_o\,\frac{\partial \ln p^{t-1}_*}{\partial x} + \frac{\partial \phi_*}{\partial x}\right]$$

B. $$b = v^{t-1} - \Delta t m_x m_y\left[u\frac{\partial}{\partial x}\left(\frac{v}{m_y}\right) + v\frac{\partial}{\partial y}\left(\frac{v}{m_x}\right)\right] - \Delta t\left(\dot{\sigma}\frac{\partial v}{\partial \sigma} + fu + F_y\right)$$

$$-\Delta t m_y\left[RT'\frac{\partial \ln p_*}{\partial y} + \frac{\partial \phi_*}{\partial y} + RT_o\frac{\partial \ln p_*^{t-1}}{\partial y}\right] \tag{3}$$

C. $$c = T^{t-1} - \Delta t\left[m_x\, u\frac{\partial T}{\partial x} + m_y\, v\frac{\partial T}{\partial y}\right] + \Delta t\ T\kappa\left[m_x\, u\frac{\partial \ln p_*}{\partial x}\right.$$

$$\left. + m_y\, v\frac{\partial \ln p_*}{\partial y}\right] + \Delta t\left(\frac{T'\kappa}{\sigma} - \frac{\partial T'}{\partial \sigma}\right)W + \Delta t\left(\sigma\frac{\partial T'}{\partial \sigma}\ W_*\right) + \Delta t\ H,$$

where variables without superscripts are to be evaluated at time t. We must still specify a vertical differencing scheme; horizontal differencing can be second or fourth order as desired. If we stagger the vertical so that W is specified at the σ levels or interfaces (to be designated $\hat{\sigma}$) as opposed to all other variables which are carried at the σ layers, the vertical derivatives, except possibly those of W, should have obvious finite depictions.

If the divergence (i.e., the second term on the left-hand side) in (2)d is replaced by the differentiated linear pressure gradient terms on the left-hand sides and the non-linear terms on the right-hand sides of (2)a and (2)b, we arrive at the following relationship:

$$(\Delta t)^{-1}\frac{\partial \bar{W}^{\tau}}{\partial \sigma} - \nabla^2(\bar{\phi}^{\tau} - \phi_{*}) - \nabla^2(RT_0 \Delta t \bar{W}_{*}^{\tau}) = -m_x m_y (\Delta t)^{-1}\left[\frac{\partial\left(\frac{a}{m_y}\right)}{\partial x} + \frac{\partial\left(\frac{b}{m_x}\right)}{\partial y}\right]$$

$$- \frac{m_x m_y}{\Delta t}\left[\frac{u}{m_y}\frac{\partial \ln p_{*}}{\partial x} + \frac{v}{m_x}\frac{\partial \ln p_{*}}{\partial y}\right] \quad (4)$$

The last term on the left-hand side comes from a combination of the second derivatives of ln p_{*} at times t and t-Δt and substituting the aforementioned finite-difference approximation to $\bar{W}_{*}^{\tau}$, i.e., $\Delta t \bar{W}_{*}^{\tau} = \ln \bar{p}_{*}^{\tau} - \ln p_{*}^{t-1}$. The ∇^2 operator is a generalized operator and is given by

$$\nabla^2 A = m_x^2 \frac{\partial^2 A}{\partial x^2} + m_x \frac{\partial m_x}{\partial x}\frac{\partial A}{\partial x} + m_y^2 \frac{\partial^2 A}{\partial y^2} + m_y \frac{\partial m_y}{\partial y}\frac{\partial A}{\partial y}$$

$$- \frac{m_x^2}{m_y}\frac{\partial m_y}{\partial x}\frac{\partial A}{\partial x} - \frac{m_y^2}{m_x}\frac{\partial m_x}{\partial y}\frac{\partial A}{\partial y},$$

where A is any variable. Note that when the mapping is conformal, i.e., $m_x = m_y$, the operator reduces to its more familiar two-term Laplacian.

After choosing a horizontal differencing scheme, vertical finite differencing will determine the discretized form of (4). Remembering that W is staggered in the vertical with respect to the other variables, $\frac{\partial W}{\partial \sigma}$ in layer k can be discretized by $(\hat{W}_{k+1} - \hat{W}_k)(\Delta\sigma_k)^{-1}$. (4) is then represented by this matrix form:

$$\mathbb{C}(\bar{W}_k^T) - \nabla^2(\bar{\Phi}_k^T - \Phi_*) - \mathbb{E}\,\nabla^2(\bar{W}_k^T) = -m_x m_y(d_k) \quad (5)$$

where the block letters are KxK matrices K being the total number of layers, and $(_k)$ represents a Kx1 vector. Here

$$\mathbb{C} = \begin{pmatrix} -r_1 & r_1 & 0 & 0 & . & . & . & . \\ 0 & -r_2 & r_2 & 0 & . & . & . & . \\ 0 & 0 & -r_3 & r_3 & 0 & . & . & . \\ . & . & . & . & . & . & . & . \\ 0 & 0 & . & . & . & . & . & -r_K \end{pmatrix}, \quad r_k = \frac{-1}{\Delta t \Delta\sigma_k}$$

$$\mathbb{E} = R\Delta t \begin{pmatrix} T_{o_1} & . & . & . & . & . & . & 0 \\ T_{o_2} & . & . & . & . & . & . & 0 \\ . & . & . & . & . & . & . & . \\ T_{o_K} & 0 & . & . & . & . & . & 0 \end{pmatrix}$$

and d_k represents the right-hand side of (4) divided by $-m_x m_y$. Note that although there are K+1 levels, (W_k) is only a vector with K components because $W_{K+1} = 0$ and need not be included.

(2)c can now render a relationship between (T_k) and (W_k), while the hydrostatic equation relates (T_k) to (Φ_k). Vertical differencing of (2)c can be accomplished in a number of ways, however, and the choice bears significant consequence. Because T_o is specified arbitrarily, one can define T_o and $\frac{\partial T_o}{\partial \sigma}$ analytically at both layers and levels. Thus, while (2)c must be specified at a layer and therefore requires some kind of averaging of the level-defined W, the coefficients of W can either be defined directly at the layers or defined at the levels and averaged along with W. Instinctively, it was felt that the differencing on the left-hand side of the equation should be consistent with the differencing employed for the right-hand side (rhs), given in (3)c. But here again a number of choices are available even for the rhs. Since W is defined at the levels while T′ is defined at the layers, we can average W over surrounding levels and use centered differencing for $\frac{\partial T'}{\partial \sigma}$. This, in effect, is the method employed by BMRC (not mentioned in the 1978 article). But this can lead to problems at the first or uppermost layer, where T′ is not defined below the first or above the top layer. BMRC, in fact, extrapolates downward to estimate a surface temperature and assumes a constant lapse rate at the top. One can avoid this questionable procedure by simply approximating the terms,

$$\left(\frac{\kappa T'}{\sigma} - \frac{\partial T'}{\partial \sigma}\right)W + \sigma \frac{\partial T'}{\partial \sigma}W.$$

for any layer k as

$$\frac{1}{2}(\kappa\ T'_k\sigma_k^{-1})(\hat{W}_{k+1} + \hat{W}_k) - \frac{1}{2}[(T'_{k+1}-T'_k)\hat{W}_{k+1}(\sigma_{k+1} - \sigma_k)^{-1} +$$

$$(T'_k - T'_{k-1})\hat{W}_k(\sigma_k-\sigma_{k-1})^{-1}] + \frac{1}{2}[(T'_{k+1} - T'_k)\hat{\sigma}_{k+1}(\sigma_{k+1} - \sigma_k)^{-1} +$$

$$(T'_k - T'_{k-1})\hat{\sigma}_k(\sigma_k - \sigma_{k-1})^{-1}]\hat{W}.$$

Now, since $\hat{W}_{K+1}$ and $\hat{\sigma}_{K+1}$ are 0, there is no need for defining a T'_{K+1}. On the other hand, $\hat{\sigma}_1 = 1$ and $\hat{W}_1 = \hat{W}$. so that the second of the bracketed terms will cancel when k=1 and there is no need to define T'_0. Representation of the first term, given analytically as $\kappa T'\sigma^{-1}W$, is also possible in more than one way because σ can be defined either at the level or layers. Thus the first term may also be written $\frac{1}{2}(\kappa\ T'_k)\left(\frac{\hat{W}_{k+1}}{\hat{\sigma}_{k+1}} + \frac{\hat{W}_k}{\hat{\sigma}_k}\right)$As we shall see, the choice of this representation can have a profound effect on the results of the model.

With any of the discretization specified above, (2)c can be written in generalized matrix form as

$$(\bar{T}'^{\tau}_k) - \mathbf{A}(\bar{W}'^{\tau}_k) = (c_k) \tag{6}$$

where the matrix $\mathbf{A}$ is defined as a KxK matrix with elements

$$\mathbf{A} = \frac{\Delta t}{2}\begin{vmatrix} G_1+s_1^1 & s_1^2 & 0 & 0 & 0 & . & . & . & 0 \\ G_2 & s_2^1 & s_2^2 & 0 & 0 & . & . & . & 0 \\ G_3 & . & s_3^1 & s_3^2 & 0 & . & . & . & 0 \\ & . & . & . & . & . & . & . & . \\ G_K & . & . & . & . & . & . & . & s_K \end{vmatrix}$$

where, for example, if we take the non-centered form over the layers, then

$$G_k = \hat{\sigma}_{k+1}(T_{o_{k+1}} - T_{oK})(\sigma_{k+1}-\sigma_k)^{-1} + \hat{\sigma}_k(T_{o_k} - T_{o_{k-1}})(\sigma_k - \sigma_{k-1})^{-1}$$

$$S_k^1 = \frac{\kappa T_{ok}}{\sigma_k} - (T_{o_k} - T_{o_{k-1}})(\sigma_k - \sigma_{k-1})^{-1}$$

$$S_k^2 = \frac{\kappa\ T_{ok}}{\sigma_k} - (T_{ok+1} - T_{ok})(\sigma_{k+1} - \sigma_k)^{-1}$$

The hydrostatic equation in matrix form can be represented as

$$(\bar{\Phi}_k^{\tau} - \Phi_*) = \mathbf{B}(\bar{T}_k^{\tau}) \qquad (7)$$

where the matrix $\mathbf{B}$ is dependent on the particular discretized relationship one chooses between layer heights and temperature. One can now substitute for $(\bar{\Phi}_k^{\tau})$ and $(\bar{W}_k^{\tau})$ in (5) using (6) and (7), and arrive at one equation for $(\bar{T}_k)$, i.e.,

$$(\mathbf{B} + \mathbf{E}\mathbf{A}^{-1})\nabla^2(\bar{T}_k^\tau) - \mathbf{C}\mathbf{A}^{-1}(\bar{T}_k^\tau) = m_x m_y (d_k) + \mathbf{E}\mathbf{A}^{-1}\nabla^2(c_k) - \mathbf{C}\mathbf{A}^{-1}(c_k) \quad (8)$$

By multiplying all terms by $(\mathbf{B} + \mathbf{E}\mathbf{A}^{-1})^{-1}$ and combining all terms on the right-hand-side, one remains with a classical Helmholtz equation in three-dimensions,

$$\nabla^2(\bar{T}_k^\tau) - \mathbf{G}(\bar{T}_k^\tau) = (R_k) \quad (8a)$$

Solution of (8a) first requires reduction to a two-dimensional problem by projecting the vectors onto each of the eigenvectors of the matrix $\mathbf{G}$. This is accomplished by substituting for $(\bar{T}_k^\tau)$ and (R_k) by (τ_k) and (Γ_k), where the latter are the former times the inverse matrix of eigenvectors of $\mathbf{G}$, i.e.,

$$(T_k) = \mathbf{V}(\tau_k),$$

where $\mathbf{V}$ is the matrix whose columns are the eigenvectors of $\mathbf{G}$. (8a) then becomes, after multiplying by $\mathbf{V}^{-1}$,

$$\nabla^2(\tau_k) - \mathbf{\Lambda}(\tau_k) = (\Gamma_k),$$

where $\mathbf{\Lambda}$ is the diagonal matrix of eigenvalues corresponding to the eigenvectors. This equation neatly divides into K two-dimensional Helmholtz equations which can normally be solved by both direct and indirect methods.

However, because the map factors appearing in the Laplacian operator are spatially dependent, only indirect iterative methods of solution can be invoked.

After τ is determined, $\bar{T}^{\tau}$ is found by multiplying through by the eigenvectors. Once $\bar{T}^{\tau}$ is known, one can theoretically invert (6) to derive $(\bar{W}_k^{\tau})$ and hence $\ln \bar{p}_{\bullet}^{\tau}$, derive $(\bar{\Phi}_k^{\tau})$ from (7) and, finally, $\bar{u}^{\tau}$ and $\bar{v}^{\tau}$ from (2)a and (2)b. This is the procedure suggested by BMRC, but there seem to be some latent problems associated with this algorithm, as will be described in the next section.

4. Deriving pressure from temperature: some pitfalls

To derive $\bar{W}^{\tau}$ from $\bar{T}^{\tau}$ one inverts (6) to obtain

$$(\bar{W}_k^{\tau}) = \Lambda^{-1}(\bar{T}_k^{\tau} - c_k) \qquad (9)$$

The updated surface pressure can then be derived by invoking the finite-difference approximation mentioned earlier, i.e.,

$$\Delta t \bar{W}_{\bullet}^{\tau} = \ln \bar{p}_{\bullet}^{\tau} - \ln p_{\bullet}^{t-1}.$$

At this point, we may consider what variables have yet to be solved in the system of equations (2) before stepping forward to the next time period. Methods for deriving ϕ^{t+1}, u^{t+1}, and v^{t+1} have been outlined in the previous section. But the vertical advection terms in (3)A and (3)B are given in terms of $\dot{\sigma}$, which has not been explicitly specified. In truth, $\dot{\sigma}$ was preempted by the creation of W to serve as a substitute variable. This exchange should require one to replace $\dot{\sigma}$ in the non-linear terms with W similar to (3)C. If $\dot{\sigma}$ is kept, one may regard the equation defining W in terms of $\dot{\sigma}$ as a diagnostic equation for $\dot{\sigma}$. If so, (9) is ipso facto a solution for $\dot{\sigma}$, as well. Yet BMRC rederives $\dot{\sigma}$ from the continuity equation using updated values of u and v. This unnecessary computation will not always yield the same $\dot{\sigma}$ values as would be derived from W and is not consistent with the rest of the model. However, when we attempted to derive $\dot{\sigma}$ directly from W, we found the values to be unreasonably large, as was the value of W_s which is the surface pressure tendency.

What caused these problems seems to stem from the inversion of the matrix $\mathbb{A}$ in (9) to obtain $(\bar{W}_k^T)$. For the sake of demonstration, we may choose a constant T_o that does not vary with height without any loss of generality. In fact, the nature of $\mathbb{A}$ and its inverse depends little on T_o, as one would expect. The values of G_k in the definition of $\mathbb{A}$ then all become zero, and $\mathbb{A}$ becomes a bidiagonal matrix with elements s_k^1 and s_k^2 equalling σ_k^{-1} or $\hat{\sigma}_k^{-1}$, depending on the choice of discretization. One would also hope that it should matter little whether we discretize the term $\kappa\ T_o\ \sigma^{-1}$ as $\kappa T_o \sigma_k^{-1}$, as $\frac{1}{2}\kappa\ T_o(\hat{\sigma}_k^{-1} + \hat{\sigma}_{k+1}^{-1})$, or as any other weighted combination of $\hat{\sigma}_k^{-1}$ and $\hat{\sigma}_{k+1}^{-1}$. But the inverse of $\mathbf{A}$ is profoundly affected by this choice. If we select the second option, i.e., $\kappa\ T_o\ W\sigma^{-1} \cong \frac{1}{2}\kappa\ T_o\left(\frac{\hat{W}_{k+1}}{\hat{\sigma}_{k+1}} + \frac{\hat{W}_k}{\hat{\sigma}_k}\right)$, then

$$\mathbf{A}^{-1} = \frac{2}{\Delta t\kappa T_o}\begin{vmatrix} \hat{\sigma}_1 & -\hat{\sigma}_1 & \hat{\sigma} & . & . & . & \pm\hat{\sigma}_1 \\ 0 & \hat{\sigma}_2 & -\hat{\sigma}_2 & . & . & . & \pm\hat{\sigma}_2 \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & . & . & \pm\hat{\sigma}_K \end{vmatrix}$$

with the last column's sign dependent on whether K is odd or even. $\bar{W}^{\tau}_{\bullet}$ depends only on the first row, which, because $\hat{\sigma}_1 = 1$, consists merely of alternate entries of ± 1. Thus, the pressure tendency will depend exclusively on the magnitude and pairing of the differences between $\bar{T}^{\tau}$ and c, the non-linear tendency of the temperature including physical processes. For pressure tendencies to remain reasonable, one must require that $2\Delta t\bar{W}^{\tau}_{\bullet}$ be of the order 10^{-4}. Since $(\kappa T_o)^{-1}$ is of the order of 10^{-2}, $\bar{T}^{\tau}$ - c should be close to or less than 10^{-2}°C in order for the pressure tendency to remain reasonable. Note that there is no apparent dependency of surface pressure tendency on Δt as $2\Delta t\bar{W}^{\tau}_{\bullet}$ cancels the $\frac{1}{\Delta t}$ in the coefficient of $\mathbf{A}^{-1}$. However, Δt is a factor in c, as given by (3)C, underlining the fact that the time differencing is not fully implicit, so that the time step still affects the non-linear tendency but not the linear tendency. Because c contains physical processes, it is unreasonable to expect that $\bar{T}^{\tau}-c$ be constrained to be less than 10^{-2}°C at all times and it is certainly unreasonable to expect that there be any specific coupling between layers so that the effects of $\bar{T}^{\tau}-c$ cancel each other. This is especially true near the surface where condensation and boundary layer processes affect the first layer temperature more than other layers.

If the first finite differencing is invoked, i.e., $\frac{1}{2}\kappa\, T_0\, \sigma_k^{-1}(\hat{W}_{k+1} + \hat{W}_k)$, then $\mathbf{A}^{-1}$ becomes

$$\frac{2}{\kappa T_0 \Delta t}\begin{pmatrix} \sigma_1 & -\sigma_2 & \sigma_3 & -\sigma_4 & . & . & . & \sim\sigma_K \\ 0 & \sigma_2 & \sigma_3 & \sigma_4 & . & . & . & \sim\sigma_K \\ . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & . & . & . & \sim\sigma_K \end{pmatrix}$$

Here, as the σ values decrease with increasing k, the stability can be maintained even with higher values of $\bar{T}^{\tau}-c$ at upper layers. This would point to a vertical discretization with higher resolution near the bottom and less resolution near the top of the model. Thus, by what seems like a simple assumption with respect to vertical finite differencing, appreciable consequences with regard to stability can occur. This would point to an extreme sensitivity on the part of the matrix $\mathbf{A}$. This sensitivity suggests an ill-conditioned situation where small differences in data are magnified to levels far beyond what one would expect.

As a simple example consider this 12-layer profile of $\bar{T}^{\tau}$-c taken from a successful forecast by RLAM during February, 1979:

$$(\bar{T}_k^{\tau} - c_k) = (-4.2932 \times 10^{-2},\ -9.8038 \times 10^{-2},\ -.105177,\ -4.1259 \times 10^{-2},$$
$$2.4924 \times 10^{-2},\ 7.1272 \times 10^{-3},\ 2.1306 \times 10^{-2},$$
$$-7.9304 \times 10^{-2},\ -2.7754 \times 10^{-2},\ -.17636,\ .38922)$$

If we assume, as BMRC did, that $\Delta\sigma = \frac{1}{12}$ and $\sigma_k = \frac{1}{2}(\hat{\sigma}_k + \hat{\sigma}_{k+1})$, and that $W\sigma^{-1}$ is given by averaging level W and dividing by layer σ one obtains

$$(\bar{W}_k^{\tau}) = (-6.038 \times 10^{-7},\ -9.967 \times 10^{-7},\ -2.340 \times 10^{-6},\ -8.988 \times 10^{-7},$$
$$-2.381 \times 10^{-7},\ -1.389 \times 10^{-7},\ -7.609 \times 10^{-7},$$
$$-1.488 \times 10^{-6},\ 6.309 \times 10^{-7})$$

The first value is the tendency of ln p_* and is of acceptable magnitude. When the averaged $\hat{\sigma}$ were used, however, $(\bar{W}_k^{\tau})$ became

$$(-1.518 \times 10^{-5},\ 1.238 \times 10^{-5},\ -1.444 \times 10^{-5},\ 9.923 \times 10^{-6},$$
$$9.223 \times 10^{-6},\ -7.421 \times 10^{-6},\ 6.300 \times 10^{-6},\ -4.763 \times 10^{-6},$$
$$2.801 \times 10^{-6},\ -3.667 \times 10^{-6},\ 1.262 \times 10^{-6})$$

This increase by two orders of magnitude of the surface pressure tendency yields values too large for stability and graphically portrays the sensitivity of inverting the $\mathbf{A}$ matrix. The same occurred when the NMC 12-layer structure was assumed. Here $\Delta\sigma$ was highly resolved near the surface, less resolved in the mid-troposhpere and highly resolved near the tropopause. The relationship of σ to $\hat{\sigma}$ is $\sigma_k^{\kappa} = (\hat{\sigma}_k^{(1+\kappa)} - \hat{\sigma}_{k+1}^{(1+\kappa)})/[(1+\kappa)(\hat{\sigma}_k - \hat{\sigma}_{k+1})]$. When σ was used $\bar{W}^{\tau}$ was -3.896×10^{-7}, while averaged $\hat{\sigma}$ values resulted in -1.518×10^{-5}, again a two order-of-magnitude increase, albeit not as large as with the BMRC discretization.

5. Conclusion

Apparently, the BMRC scheme can lead to instabilities wrought by the vertical differencing and the necessity to derive pressure tendencies from the vertical profile of the temperature tendency minus the non-linear terms of the thermodynamic equation. This linkage, although theoretically sound, depends inordinately on the differencing scheme, demonstrating that the $\mathbf{A}$ matrix may be ill-conditioned and unreliable as a means for deriving surface pressure and vertical velocity. Inverting the problem is a possibility. It involves substituting in (5) for ϕ in terms of W by means of (7) and (9) and solving for $\bar{W}^{\tau}$ then deriving $\bar{T}^{\tau}$. Unfortunately, that procedure does not solve the problem, first, because (9) is still used in the substitution and second, because values for W at the lateral boundaries are hard to come by since vertical velocity is normally derived diagnostically from forecast divergence and is not stored as a prognostic variable. Attempts at forecasting $\bar{W}^{\tau}$ and deriving $\bar{T}^{\tau}$ did not seem any more satisfactory.

Despite its failings, the BMRC semi-implicit scheme does have its application. If time steps are constrained, the scheme could produce forecasts still more economically than explicit schemes that must keep their time steps in proportion to the grid size. The problem is knowing ahead of time how large a time step the scheme can tolerate. Several RLAM forecasts were produced over North America and the North Sea with the semi-implicit scheme. These forecasts contained physical parameterizations including boundary layer fluxes and large-scale and convective precipitation with a resolution of approximately 200 km over North America and about 100 km over the North Sea. The semi-implicit scheme yielded forecasts with time steps of about 600 s for both locations and these forecasts compared well with the Brown-Campana explicit formulation using time steps of 180 s for North America and 120 s for the North Sea. Near the lateral boundaries, the North American forecasts seemed to be noisier than the Brown-Campana forecasts. But even with a time step of 600 s, one forecast over the North Sea failed after successfully forecasting for close to 40 h of simulated time. When the forecast was rerun with a time step of 480 s, it was stable to 48 h. This emphasizes the unreliability of the scheme and serves as an illustration of what can go wrong without warning because of an unexpected shock to the model.

The failure of the BMRC scheme to provide stable results is not necessarily the result of a flaw endemic to all semi-implicit schemes. As mentioned, the semi-implicit scheme of the AFGL gsm or of the Canadian model are apparently not subject to the same difficulties as the BMRC model. It may even be possible to slightly modify the BMRC scheme to make it less sensitive. Results from this study, however, should encourage close scrutiny of any numerical scheme before adaptation to any model.

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